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Nineteenth century roots of quasideterminants

Francine F. Abeles

Departments of Mathematics and Computer Science, Kean University, Union, NJ 07083, United States

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ABSTRACT

In this expository paper I provide a complete record of the nineteenth century publications that bear on the development of quasideterminants in the twentieth century. Two important recursive feasible algorithms, Sylvester's from 1851 and Dodgson's from 1866 are discussed, and the antecedents of both are traced back to work by Jacobi.

1. Introduction

In a landmark paper published in 1991 I.M. Gelfand and V.S. Retakh constructed general quasideterminants of matrices over noncommutative rings. Since then many papers on the subject have appeared, most of them citing primary or secondary historical sources vaguely, incorrectly, or not at all. One of the aims of this paper is to accurately identify these nineteenth century publications that bear on general quasideterminants. The second aim is to consider the standard definitions of determinants over commutative rings and of quasideterminants with the goal of identifying expressions that will apply naturally to both. The third aim is to analyze two nineteenth century papers that originally concerned determinants, James J. Sylvester's paper from 1851, and Charles L. Dodgson's paper from 1866, that subsequently have been shown to have noncommutative analogues. The final aim is to identify the antecedents of Sylvester's and Dodgson's papers in earlier work of Carl G.J. Jacobi.

E-mail address: fabeles@kean.edu

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2. Early publications explicitly on quaternion determinants

In a paper published in the *Philosophical Magazine* in 1845, Arthur Cayley discussed some recent results that William Rowan Hamilton had published on quaternions. Cayley stated that "The ordinary definition of a determinant leads to that of a quaternion determinant" [6, p. 143]. However, the ordinary definition is not a canonical one because the expansion can be, for example, along the first row (row-determinant), or along the first column (column-determinant). Contributing to some of the confusion in citations of Cayley's paper is that Cayley called a column a vertical row. But this is the first reference to the determinant of a matrix whose entries come from a division ring rather than from a field. Contrary to what some authors have claimed, Cayley offered no further suggestions about how to define such a determinant.

Another paper relating to quaternion determinants appeared in 1896 by Charles Jasper Joly that was published in the *Proceedings of the Royal Irish Academy*, titled, "Quaternion invariants of linear vector functions and quaternion determinants." Joly, too, did not offer a new definition for a quaternion determinant. He wrote, "[a] determinant whose constituents are quaternions is unmeaning until some convention is adopted respecting its expansion. If ... the order of the constituents in the expansion shall follow the order of the rows, all indefiniteness is removed" [15, p. 1]. As far as I know, his paper has not been cited in any of the historical material appearing in modern papers on the origins of quasideterminants. Joly also edited the second edition of Hamilton's *Elements of Quaternions*, the first volume appearing in 1899 and a year later published his own *Supplement to oeuvres of Hamilton.* A long abstract of a paper read before a meeting of the American Mathematical Society on February 25, 1899 by James Mills Peirce, "Determinants of Quaternions" was published in the Society's Bulletin in that year. Peirce added nothing more than was already known: "I adopt ... the convention that the order of factors in every term of a determinant is the same as the order of columns in the matrix" [19, p. 335]. For a valuable survey of definitions of quaternion matrices and resulting developments in the century following Cayley, see [5]. The subject of noncommutative determinants did not appear again until the early part of the twentieth century when J.H.M. Wedderburn constructed a theory of determinants of matrices with general noncommutative entries [22].

3. General quasideterminants

We begin with Gelfand and Retakh's definition of a quasideterminant [9, p. 92]. Let R be a ring with unity; let $A = (a_{ij}),\ i, j = 1, \dots, n$ be a matrix over R. A^{pq} denotes the matrix of order $n - 1$ obtained from A by deleting the pth row and qth column.

Let $\xi_{p,q} = (a_{p1}, \dots, \hat{a}_{pq}, \dots, a_{pn})$; $\eta_{p,q} = (a_{1q}, \dots, \hat{a}_{pq}, \dots, a_{nq})$. The quasideterminant of index pq is given by:

$$|A|_{pq} = a_{pq} - \xi_{p,q}\,(A^{pq})^{-1}\eta_{p,q}$$

provided that $(A^{pq})^{-1}$ is defined. Note that $\xi_{p,q}$ is a matrix of one row, $\eta_{p,q}$ is a matrix of one column, and the multiplication of the three terms on the right hand side is matrix multiplication. From the definition we can see that n^2 quasideterminants are defined but all of them may not exist. When $n = 2$, we obtain:

$$|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \qquad |A|_{21} = a_{21} - a_{22}a_{12}^{-1}a_{11},$$
$$|A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22}, \qquad |A|_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}.$$

When $n = 3$, the definition produces nine terms each one involving two nested inversions which are considerably more difficult to deal with. For $n > 3$, the situation becomes far too complex to handle.

We immediately see that quasideterminants are defined differently from determinants. The definition of a determinant of a matrix as a polynomial function of its entries does not apply to quasideterminants. Daniel Krob and Bernard Leclerc note that the quasideterminants $|M|_{pq}$ of a matrix $M = (m_{ij})$ whose entries are from a field K are obtained by applying to the rational expressions $|A|_{pq}$, the specialization $a_{ij} \to m_{ij}$. $|M|_{pq}$ will be well defined if M^{pq}, the matrix formed by deleting the pth row and qth column of M, is invertible in K. And if K is a commutative field, $|M|_{pq} = (-1)^{p+q} \det M / \det M^{pq}$. As defined above, we see that quasideterminants can be considered noncommutative analogues of the ratio of a determinant to one of its principal minors [9, p. 92;17, p. 5].

4. Sylvester's identity from 1851

Much of the confusion surrounding the correct citation of Sylvester's algorithm for computing both determinants and quasideterminants arises from two sources. First, Sylvester published three papers that year, and all of them appeared in different issues of the same journal. Secondly, in his paper, he did not include a proof of his algorithm. The relevant paper is 21. Sylvester called his theorem "remarkable," and gave it as an example of what he described as his new, simple, powerful, and natural umbral or bilitereral notation [21, pp. 296–297]. The generalization of this identity to entries from a noncommutative ring was a key discovery made by Gelfand and Retakh in 1991.

A modern statement of Sylvester's identity by Akritas et al. [4] is: Let

$A = (a_{ij}),\ i, j = 1, 2, \ldots, n$ be an $n \times n$ matrix. Let a_{ij}^k, $1 \leqslant k \leqslant n$, $k \leqslant i, j \leqslant n$ be a determinant of order k obtained from A by adding the ith row and jth column to the upper left corner minor of order $k-1$. Setting $a_{00}^0 = 1$, then for $1 \leqslant p \leqslant n$, we have

$$\det D_p(A) = (\det A) \cdot \left(a_{p-1,p-1}^{p-1}\right)^{n-p},$$

where $D_p(A)$ is of order $n - p + 1$, and its elements have the form a_{ij}^p where $p, i, j = 1, 2, \ldots, n$ are minors of A of order p. Note that $(a_{ij}^1) = (a_{ij})$.

The importance of Sylvester's identity is that it provides a recursive method for computing determinants or quasideterminants by reducing the computation of a determinant (quasideterminant) of order n to the computation of a determinant (quasideterminant) of order $n-1$ whose $(n-1)^2$ entries are determinants (quasideterminants) of order 2.

Krob and Leclerc note that the identity can be deduced directly from the relation

$$B_{RP} = (A_{PR} - A_{PS} A_{QS}^{-1} A_{QR})^{-1},$$

where A_{PQ} denotes the submatrix with row indices from P; column indices from Q; P, Q are subsets of {1,...,n}; and A and $A^{-1} = B$ are partitioned into blocks as follows:

$$A = \begin{pmatrix} A_{PR} & A_{PS} \\ A_{QR} & A_{QS} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{RP} & B_{RQ} \\ B_{SP} & B_{SQ} \end{pmatrix}.$$

Assuming that $|P| = |R|$ and $|Q| = |S|$ so that the block matrices are square, block multiplication yields four relations of which B_{RP} is one [17, p. 5, 11].

5. Dodgson's identity from 1866

Gelfand et al. [11] discuss Sylvester's identity in their 2005 paper. For a matrix $A = (a_{ij}),\ i, j = 1, 2, \ldots, n$ over a commutative ring and an invertible submatrix $A_0 = (a_{ij}),\ i, j = 1, \ldots, k$ of A, then for $p, q = k+1, \ldots, n$, set

$$\hat{C}_{pq} = \det \begin{pmatrix} & & & a_{1q} \\ A_0 & & & \cdot \\ & & & \cdot \\ & & & a_{kq} \\ a_{p1} & \ldots & a_{pk} & a_{pq} \end{pmatrix}_{pq},$$

$\widehat{C} = (\hat{c}_{pq}),\ p, q = k+1, \ldots, n$. Then $\det A = \det \widehat{C} / (\det A_0)^{n-k-1}$.

They remark that their *noncommutative* version of Sylvester's identity applies to the case where A_0 is a square submatrix of A composed of some rows and columns of A that are not necessarily consecutive and not necessarily the same rows and columns of A. When $A_0 = (a_{ij}),\ i, j = 2, \ldots, n-1$, this generalization, first reported in 1997, "is an analogue of a well-known commutative identity ... the 'Lewis Carroll identity"' [11, p. 18;10, p. 10]. This is the first time that Dodgson's identity has been identified as a special case of Sylvester's identity.

Dodgson's identity appeared in *Proceedings of the Royal Society* [7]. We state the identity in modern terms in the following way. For an n by n matrix A, let $A_r(i, j)$ denote the r by r minor consisting of r contiguous rows and columns of A, beginning with row i, column j. Note that $A_n(1, 1) = \det A$, $A_{n-2}(2, 2)$ is the central minor; $A_{n-1}(1, 1), A_{n-1}(2, 2), A_{n-1}(1, 2), A_{n-1}(2, 1)$ are the northwest, southeast, northeast, and southwest minors, respectively. Then

$$A_{n-2}(2, 2)A_n(1, 1) = A_{n-1}(1, 1)A_{n-1}(2, 2) - A_{n-1}(1, 2)A_{n-1}(2, 1).$$

It is easy to see that the determinant of A can be expressed as the ratio of the difference of the products of two pairs of minors to the central minor. This ratio is a rational function of all its connected minors of any two consecutive sizes [1, pp. 331–334; 3, pp. 3–4].

Dodgson recognized the problems that would be caused by the occurrence of zeroes in the interior of any derived block. Nevertheless, he believed "[i]t will be found in practice that ... the whole amount of labour will still be much less than that involved in the old process of computation" [8, p. 122].

Like Sylvester's identity, Dodgson's identity provides a recursive method for computing both determinants and quasideterminants by reducing the computation of that determinant or quasideterminant to the computation of 2×2 determinants or quasideterminants, respectively. As we have indicated, the computation of quasideterminants for very large matrices is even more tedious than for determinants. Both Sylvester's and Dodgson's identities can be programmed to run in $O(n^3)$ time, i.e. in cubic polynomial time and that time can be improved considerably when they are run on parallel processors.

Another important aspect of Sylvester's and Dodgson's identities is that they both deal with block matrices, i.e. with the decomposition of a square matrix A into its submatrices, A_{ij}. Sylvester first gave them the name, "compound matrices," in 1850 in connection with determinants whose elements themselves are determinants. If these blocks are considered to be the entries of a matrix X, then the quasideterminant of X will be another matrix C and its quasideterminant will be equal to a suitable quasideterminant of A. In their 1997 paper, Gelfand and Retakh called this important property the "heredity principle." It does not hold for determinants since determinants of block matrices are not defined.

Of all the papers on quasideterminants only the 1997 and 2005 papers by Gelfand et al. include Dodgson's identity as an additional algorithm for computing them. The most probable reasons for its omission in the many other papers on quasideterminants that have appeared in the last twenty years are that Dodgson's mathematical contributions generally are not nearly as well known as Sylvester's and Dodgson's important algorithm emerged from relative obscurity only in the last fourteen years of the twentieth century.

6. Antecedents of Sylvester's and Dodgson's identities

In 1841 Jacobi [13,14] worked on a special type of compound determinant that he called a functional determinant. His second paper from 1841, here referred to as J2, "De determinantibus functionalibus" in *Crelle's Journal* is the antecedent of Sylvester's identity. Jacobi wrote three papers on determinants that appeared in *Crelle's Journal* in the same volume which is probably the source of error in the citations (or lack thereof) of his paper in connection with Sylvester's identity. Sylvester did not refer to Jacobi in his paper of 1851.

In 1909, G. Kowalewski used J2 to prove Sylvester's theorem in his *Einführung in die Determinantentheorie* [16, pp. 90–93]. A modern statement of J2 by Akritaset al. [4] is: If $\det A \neq 0,\ \det(\operatorname{adj} A)^p$ is a minor of the $\det(\operatorname{adj} A)$ of order p, $\tilde{a}_{rs}^p$ is the corresponding minor of $\det A$, then $\det(\operatorname{adj} \widetilde{A})^p$ differs from the algebraic complement of $\tilde{a}_{rs}^p$ by the factor $(\det A)^{p-1}$. Note that $\operatorname{adj} A$, the *adjugate* of A, also known as the *adjoint* of A, is the $n \times n$ matrix whose (i, j) entry $(\operatorname{adj} A)_{ij}$ is $(-1)^{i+j} \det A_{ji} = \alpha_{ij}$, and α_{ji} is the *algebraic complement* of α_{ij}.

Jacobi's paper from 1833 [12] in the same journal, "De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium mutiplicium" is the antecedent of Dodgson's identity. Section 11 of his earlier 1841 paper in *Crelle's Journal*, here referred to as J1, "De formatione et proprietatibus Determinantium" contains a proof of a general theorem that includes the theorem he had proved in his 1833 paper.

Using modern terminology, Adrian Rice and Eve Torrence proved Jacobi's 1833 theorem: If A is an $n \times n$ matrix; $[A_{ij}]$ is an $m \times m$ minor of A, $m < n$; $[A'_{ij}]$ is the corresponding $m \times m$ minor of A', and $[A^*_{ij}]$ is the complementary $(n-m) \times (n-m)$ minor of A then,

$$\det[A'_{ij}] = (\det A)^{m-1} \ \det[A^*_{ij}].$$

Note that for an $n \times n$ matrix, and any $(n-m) \times (n-m)$ minor of it, the complementary $m \times m$ minor is the $m \times m$ matrix diagonally adjacent to that minor [20, p. 90-1].

Dodgson's identity is a special case of this theorem when $m = 2$. Dodgson did not refer to Jacobi in his paper of 1866 when he stated the theorem and proved it for this case. The first implied reference to Jacobi's theorem appears in a reply letter to Dodgson dated 2 April 1866 from William Spottiswoode, the author of *Elementary Theorems relating to Determinants*, where he wrote, "The Theorem upon which it [condensation] is founded is, as you are doubtless aware, known" [2, p. 170]. And Dodgson makes no mention of this theorem in his book published the following year, *An Elementary Treatise on Determinants*, where he also states the theorem and proves his identity as Proposition 7 in Chapter 2 [8, pp. 25–30].

7. Conclusion

General quasideterminants are an important topic currently in noncommutative algebra. Although they are not matrix invariants, they do satisfy many of the classical identities, suitably modified, like Cramer's rule and Muir's law of extensionality among many others [18]. Sylvester's and Dodgson's algorithms are feasible recursive methods to compute both determinants and quasideterminants. And from a foundational perspective, both determinants and quasideterminants can be expressed in terms of ratios, i. e. quasideterminants are noncommutative analogues of the ratio of the determinant of an $n \times n$ matrix to the determinant of a suitable $(n-1) \times (n-1)$ submatrix.

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